The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens** !

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L) I(z)$$

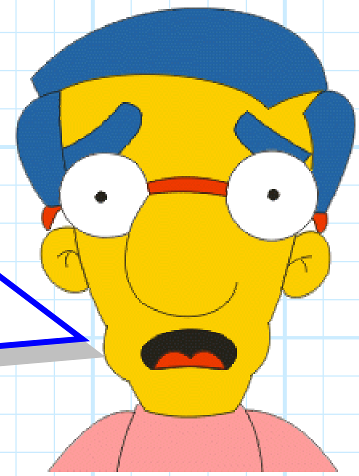
Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z} \right)$$

Q: ***But wait***! *I thought we already knew current I(z). Isn't it:*

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for I(z) be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \qquad \text{and} \qquad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{\gamma}{R + j\omega L}\right) V^+(z) \qquad \text{and} \qquad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We find that:

$$\frac{\gamma}{R+j\omega L}=\frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{R+j\omega L}=\sqrt{\frac{G+j\omega C}{R+j\omega L}}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^{+}(z)}{I^{+}(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad \text{and} \quad \frac{-V^{-}(z)}{I^{-}(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

Q: *What's so startling about **this** conclusion?*

A: Note that although the magnitude and phase of each propagating wave is a **function** of transmission line **position** z (e.g., $V^{+}(z)$ and $I^{+}(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the transmission line **only** (*R, L, G, C*).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance** Z_0.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

Note that instead of characterizing a transmission line with **real** parameters R, G, L, and C, we can (and typically do!) describe a transmission line using **complex** parameters Z_0 and γ.